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Erdős–Turán Type Theorems on Quasiconformal Curves and Arcs

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The theorems of Erdős and Turán mentioned in the title are concerned with the distribution of zeros of a monic polynomial with known uniform norm along the unit interval or the unit disk. Recently, Blatt and Grothmann (*Const. Approx.* **7** (1991), 19–47), Grothmann ("Interpolation Points and Zeros of Polynomials in Approximation Theory," Habilitationsschrift, Katholische Universität Eichstätt, 1992), and Andrievskii and Blatt (*J. Approx. Theory* **88** (1977), 109–134) established corresponding results for polynomials, considered on a system of sufficiently smooth Jordan curves and arcs or piecewise smooth curves and arcs. We extend some of these results to polynomials with known uniform norm along an arbitrary quasiconformal curve or arc. As applications, estimates for the distribution of the zeros of best uniform approximants, values of orthogonal polynomials, and zeros of Bieberbach polynomials and their derivatives are obtained. We also give a negative answer to one conjecture of Eiermann and Stahl ("Zeros of orthogonal polynomials on regular N-gons," *in* Lecture Notes in Math. **1574** (1994), 187–189). © 1999 Academic Press

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1. INTRODUCTION

Erdős and Turán [15, 16] established some quantitative results on the distribution of the zeros of a monic polynomial, for which an upper bound for the uniform norm either along the interval $[-1, 1]$ or the closed unit

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disk $\bar{D} := \{z: |z| \leqslant 1\}$ is known. The estimates obtained in [15, 16] have been improved and generalized by many other authors (for a survey, see [8, 10, 11–13, 18, 29, 30]). We will restrict our attention to the consideration of a monic polynomial on a compact set E of the complex plane $\mathbb{C}$ (instead of $[-1, 1]$ or $\bar{D}$). A survey of the most recent results in this direction can be found in [10] (where ∂E is a system of sufficiently smooth curves and arcs), [18] (∂E consists of a finite number of analytic arcs), and [8] (∂E consists of a finite number of Dini-smooth arcs).

The main purpose of this paper is to prove the same assertions for a monic polynomial with known uniform norm on a quasiconformal curve or arc [2, 22].

2. MAIN DEFINITIONS AND RESULTS

Let F be an arbitrary bounded continuum (not a single point) in the complex plane $\mathbb{C}$ with simply connected complement $\Omega := \bar{\mathbb{C}} \backslash F$, where $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, and let $L := \partial\Omega = \partial F$ be their common boundary.

Let Φ denote the Riemann function that conformally and univalently maps Ω onto the exterior $\Delta := \bar{\mathbb{C}} \backslash \bar{D}$ of the unit disk $D := \{w: |w| < 1\}$ and which is normalized by the conditions

$$\Phi(\infty) = \infty, \qquad \Phi'(\infty) := \lim_{z \to \infty} \frac{\Phi(z)}{z} := (\operatorname{cap} L)^{-1} > 0,$$

where $\operatorname{cap} F = \operatorname{cap} L$ denotes the logarithmic capacity of F (or L) [31].

Set $\Psi := \Phi^{-1}$. By $\mu = \mu_F = \mu_L$ we denote the equilibrium measure (distribution) for F [31].

Throughout what follows L will be a quasiconformal curve or arc [2, 22]. We recall that a Jordan curve or a Jordan arc L is called K-quasiconformal, $K \geqslant 1$, or, briefly, quasiconformal, if there exists a K-quasiconformal mapping of the plane onto itself which carries L into a circle or a line segment, respectively. Ahlfors (see [2]) has established a geometric criterion for quasiconformality of a curve which can be formulated as follows: A Jordan curve L is quasiconformal iff for any pair of points z_1 and $z_2 \in L$ the inequality

$$\min\{\operatorname{diam} L', \operatorname{diam} L''\} \leqslant c\, |z_1 - z_2|$$

holds with some constant $c = c(L) \leqslant 1$, where L' and L'' are the two arcs which $L \backslash \{z_1, z_2\}$ consists of.

We note that the same description is valid for a quasiconformal arc as well [28].

Using Ahlfors' criterion, one can easily verify that convex curves, curves of bounded variation without cusps, and rectifiable Jordan curves which have the same order of arc length and chord length are quasiconformal. At the same time, as Belinskii's example [9, p. 42] shows, each part of a quasiconformal curve can be nonrectifiable.

A Jordan domain bounded by a quasiconformal curve is also called a quasidisc.

Let $p = p_n \in \mathbb{P}_n$, where $\mathbb{P}_n, n \in \mathbb{N} := \{1, 2, ...\}$, denotes the set of all algebraic polynomials of degree n. We associate with p the normalized counting measure for its zeros, i.e.,

$$\nu(A) = \nu_p(A) := \frac{\text{number of zeros of } p \text{ in } A}{n} \qquad (A \subset \mathbb{C}),$$

where the zeros are counted with their multiplicities.

Consider the logarithmic potentials of the measures μ and ν

$$\begin{aligned} U^\mu(z) &:= -\int \log |z - \zeta| \, d\mu(\zeta) \\ &= \begin{cases} -\log |\Phi(z)| - \log \operatorname{cap} L & \text{if } z \in \Omega \setminus \{\infty\} \\ -\log \operatorname{cap} L & \text{if } z \in F, \end{cases} \\ U^\nu(z) &:= -\int \log |z - \zeta| \, d\nu(\zeta) = -\frac{1}{n} \log |p(z)| \qquad (z \in \mathbb{C}), \end{aligned}$$

and their difference

$$U^{\mu - \nu}(z) := U^\mu(z) - U^\nu(z) \qquad (z \in \mathbb{C}).$$

Our basic results will be formulated in terms of the quantities

$$\varepsilon_p := \sup_{z \in \mathbb{C}} U^{\mu - \nu}(z) = \frac{1}{n} \log \|p\|_L - \log \operatorname{cap} L,$$

if L is an arc, and

$$\delta_p := 2\varepsilon_p - U^{\mu - \nu}(z_0) = \frac{2}{n} \log \|p\|_L - \frac{1}{n} \log |p(z_0)| - \log \operatorname{cap} L,$$

where $z_0 \in \operatorname{int} L$ is an arbitrary fixed point, if L is a curve.

Here and in what follows, the symbol $\|\cdot\|_A$ denotes the supremum norm over the set $A \subset \mathbb{C}$ and $\operatorname{int} L$ denotes the collection of points interior to the Jordan curve L. In addition, set $\operatorname{ext} L := \overline{\mathbb{C}} \setminus \overline{\operatorname{int} L}$.

To start with, let L be a K-quasiconformal curve and let $z_0 \in G := \operatorname{int} L$ be a fixed point. Denote by $w = \varphi(z)$ the conformal mapping of G onto D with the normalization $\varphi(z_0) = 0$, $\varphi'(z_0) > 0$. Set $\psi := \varphi^{-1}$.

The functions Φ, Ψ, φ, ψ can be naturally extended to homeomorphisms between the appropriate closed domains and we keep the previous notations for these extensions.

Further, for $\zeta \in \mathbb{C}\backslash\{z_0\}$ set

$$\zeta_L := \begin{cases} \Psi\left(\dfrac{\Phi(\zeta)}{|\Phi(\zeta)|}\right) & \text{if} \quad \zeta \in \Omega\backslash\{\infty\} \\ \psi\left(\dfrac{\varphi(\zeta)}{|\varphi(\zeta)|}\right) & \text{if} \quad \zeta \in G\backslash\{z_0\} \\ \zeta & \text{if} \quad \zeta \in L, \end{cases}$$

$$L_r^+ := \{\zeta\colon |\Phi(\zeta)| = 1 + r\} \qquad (r \geqslant 0),$$

$$L_r^- := \{\zeta\colon |\varphi(\zeta)| = 1 - r\} \qquad (0 \leqslant r < 1).$$

Let J be an arbitrary subarc of L. For $\sigma > 0$ and $0 < \tau < 1$ define

$$J_\sigma^+ := \{\zeta \in L_\sigma^+ : \zeta_L \in J\},$$

$$J_\tau^- := \{\zeta \in L_\tau^- : \zeta_L \in J\},$$

$$E_{\sigma,\tau} := (\operatorname{ext} L_\tau^-) \cap (\operatorname{int} L_\sigma^+),$$

$$A_{\sigma,\tau}(J) := \{\zeta \in E_{\sigma,\tau} \colon \zeta_L \in J\}.$$

Consider the function $\Phi \circ \psi$ (given on the unit circle ∂D) and its modulus of continuity

$$\omega_{\Phi\circ\psi}(x) := \sup_{\substack{|w| = |t| = 1,\\ |w-t| \leqslant x}} |(\Phi\circ\psi)(w) - (\Phi\circ\psi)(t)| \qquad (x > 0).$$

The functions Φ and ψ satisfy a Hölder condition. The validity of this well-known fact follows, for example, from Lemma 1 below.

Hence,

$$\omega_{\Phi\circ\psi}(x) \leqslant Cx^\alpha \qquad (x > 0) \tag{2.1}$$

with some constants $C > 0$ and $0 < \alpha \leqslant 1$.

More precise information about the connection between the geometry of L and the exponent α in (2.1) can be derived, for example, from [23, 24]. We only state the following remark concerning the case of a piecewise smooth curve L.

Following [27], a smooth Jordan curve L is called Dini-smooth if the angle $\beta(s)$ of the tangent, considered as a function of the arc length s, satisfies

$$|\beta(s_2)-\beta(s_1)|<h(s_2-s_1) \qquad (s_1<s_2),$$

where $h(x)$ is an increasing function for which

$$\int_0^1 \frac{h(x)}{x}\,dx<\infty.$$

We call a Jordan arc Dini-smooth if it is a subarc of some Dini-smooth curve.

It is simple to derive from well-known distortion properties of the conformal mappings Φ and ψ [27, Chapter 3] that if L consists of a finite number m of Dini-smooth arcs which meet under the with respect to G inner angles $\beta_j\pi$, $0<\beta_j<2$, $j=1, ..., m$, then (2.1) is valid with

$$\alpha=\min\left\{\frac{\min_{1\leqslant j\leqslant m}\beta_j}{2-\min_{1\leqslant j\leqslant m}\beta_j},\,1\right\}. \tag{2.2}$$

Note that by this formula the value of α is the same (and equal to 1) for smooth L and for piecewise smooth L with $\beta_j\geqslant 1$, $j=1, ..., m$.

THEOREM 1. *Let L be a K-quasiconformal curve and let the function $\Phi\circ\psi$ satisfy condition* (2.1). *Let $z_0\in G:=\operatorname{int} L$ and $0<c_0<1$ be fixed. Suppose that $p\in\mathbb{P}_n$, $n\in\mathbb{N}$, is a monic polynomial such that $\delta_p<1/2$. Then there exists a constant $c>0$ depending only on K, α, C, z_0, c_0 such that*

$$|(\mu_L-\nu_p)(A_{\sigma,\tau}(J))|\leqslant c\,\delta_p^{\alpha/(1+\alpha)} \tag{2.3}$$

for all subarcs J of L and all σ and τ with the property

$$\sigma\geqslant\sigma_n=\sigma_n(p,z_0,c_0,\alpha):=c_0\,\delta_p^{\alpha/(1+\alpha)},$$
$$1>\tau\geqslant\tau_n=\tau_n(p,z_0,c_0,\alpha):=\sigma_n^{1/\alpha}.$$

If F is an arc, an analogue to Theorem 1 holds as well. Namely, let $F=L$ be a K-quasiconformal arc. Denote by z_1 and z_2 the endpoints of L. Since the function Φ can be extended continuously to these points, we set for $r>0$ and $j=1, 2$,

$$t_j:=\Phi(z_j), \qquad \Delta_1:=\{t\colon |t|>1,\ \arg t_1<\arg t<\arg t_2\},$$
$$\Delta_2:=\Delta\setminus\bar{\Delta}_1, \qquad \Omega_j:=\Psi(\Delta_j), \qquad J_j:=\bar{\Delta}_j\cap\bar{D},$$
$$L_r:=\{\zeta\in\Omega\colon |\Phi(\zeta)|=1+r\}.$$

A routine category argument shows that $\partial\Delta_1$ and $\partial\Delta_2$ are both quasiconformal curves. Moreover, the curve $\partial\Omega_1=\partial\Omega_2$ is quasiconformal, too (see [4, Lemma 1]). Therefore, the restriction Φ_j, $j=1,2$, of the function Φ to the region Ω_j can be extended to a K_1-quasiconformal mapping of the extended complex plane $\overline{\mathbb{C}}$ onto itself, where $K_1=K_1(L)>1$ is a suitable constant (see [2, Chapter IV]).

Using Lemma 1 below, this fact makes it possible to obtain

$$\omega_{\Phi_1\circ\Psi_2}(x)+\omega_{\Phi_2\circ\Psi_1}(x)\leqslant Cx^{\alpha} \qquad (x>0), \tag{2.4}$$

where $C>0$ and $0<\alpha\leqslant 1$ are some constants depending only on L, $\Psi_j:=\Phi_j^{-1}$ and where $\omega_{\Phi_1\circ\Psi_2}$ as well as $\omega_{\Phi_2\circ\Psi_1}$ denote the moduli of continuity of the functions $\Phi_1\circ\Psi_2$ and $\Phi_2\circ\Psi_1$ on J_2 and J_1, respectively.

For an arbitrary subarc J of L and $\sigma>0$ set

$$E_\sigma:=\operatorname{int} L_\sigma, \qquad A_\sigma(J):=\{\zeta\in E_\sigma: \zeta_L\in J\},$$

where we use the notation

$$\zeta_L:=\Psi_j\left(\frac{\Phi_j(\zeta)}{|\Phi_j(\zeta)|}\right) \qquad (\zeta\in\bar{\Omega}_j).$$

THEOREM 2. *Let L be a quasiconformal arc satisfying condition* (2.4) *and let $c_0>0$ be fixed. Suppose that $p\in\mathbb{P}_n$, $n\in\mathbb{N}$, is a monic polynomial such that $\varepsilon_p\leqslant 1$. Then there exists a constant $c>0$ depending only on L, α, C, c_0 such that*

$$|(\mu_L-\nu_p)(A_\sigma(J))|\leqslant c\varepsilon_p^{\alpha/(1+\alpha)} \tag{2.5}$$

for all subarcs J of L and all σ with

$$\sigma\geqslant\sigma_n=\sigma_n(p,c_0,\alpha):=c_0\varepsilon_p^{1/(1+\alpha)}.$$

Roughly speaking Theorems 1 and 2 are extensions of some results in [10] to the case of curves and arcs with corners (not cusps). In fact, if $L\in C^{2+}$ as in the result of Blatt and Grothmann cited above, then $\alpha=1$ and Theorems 1 and 2 give the same estimates as [10, Theorem 2]. It may be interesting to note that according to (2.2) condition (2.1) with $\alpha=1$ is satisfied for some curves with corners, too.

The requirement of quasiconformality of the curve or arc imposed in the theorems above is essential (for details, see [8]).

In what follows we denote by $c, c_1, \ldots$ positive constants (different each time, in general) that either are absolute or depend on parameters not essential for the arguments; sometimes such a dependence will be indicated.

For positive a and b we use the expression $a \preccurlyeq b$ (order inequality) if $a \leqslant cb$ for some $c > 0$. The expression $a \asymp b$ means that $a \preccurlyeq b$ and $b \preccurlyeq a$ simultaneously.

Set

$$d(A, B) := \operatorname{dist}(A, B) := \inf_{z \in A, \zeta \in B} |z - \zeta| \qquad (A, B \subset \mathbb{C}).$$

3. APPLICATION TO ZEROS OF BEST UNIFORM APPROXIMANTS

As a direct consequence of Theorems 1 and 2, a quantitative version of a result from [12] for the zeros of the polynomials of best uniform approximation to functions on compact sets in $\mathbb{C}$ can be obtained.

THEOREM 3. *Let F be a closed quasidisc or a quasiconformal arc satisfying condition* (2.1) *or* (2.4), *respectively. Assume that the function f is analytic in the interior of F, continuous on F, and not infinitely often differentiable on the boundary of F. If we denote by p_n^* the best uniform approximation of f on F with respect to $\bigcup_{k=1}^{n} \mathbb{P}_k$, then, for all $0 < \sigma_0 < 1$, there exist an infinite sequence $\Lambda \subseteq \mathbb{N}$ such that, for any subarc J of $L := \partial F$, $n \in \mathbb{N}$, $\sigma > \sigma_0$ and $\sigma_0 < \tau < 1$,*

$$|(\mu_L - \nu_{p_n^*})(A_{\sigma, \tau}(J))| \leqslant c \left(\frac{\log n}{n} \right)^{\alpha/(1+\alpha)}$$

if L is a curve (*respectively,*

$$|(\mu_L - \nu_{p_n^*})(A_\sigma(J))| \leqslant c \left(\frac{\log n}{n} \right)^{\alpha/(1+\alpha)}$$

in the case of an arc), *where the constant $c > 0$ is independent of J, n, and σ.*

The proof of this theorem is completely analogous to the proof of [10, Corollary 4] and we shall not dwell on it.

4. APPLICATION TO THE DISTRIBUTION OF VALUES OF ORTHOGONAL POLYNOMIALS

Let L be an arbitrary quasiconformal curve, and let h be a weight function on $G := \operatorname{int} L$, i.e., positive and measurable function on G. Next, let

$Q_n(z) = Q_n(h, z) = \lambda_n z^n + \cdots$, $\lambda_n > 0$, $n \in \mathbb{N}_0 := \{0, 1, ...\}$, be a sequence of polynomials orthogonal on G with respect to the weight function h, that is,

$$\int_G Q_k(z)\, \overline{Q_l(z)}\, h(z)\, dm(z) = \begin{cases} 1 & \text{if } \ k = l \\ 0 & \text{if } \ k \neq l, \end{cases}$$

where $dm(z)$ is the 2-dimensional Lebesgue measure.

We assume here and throughout that

$$h(z) \geqslant c_1 (d(z, L))^m \qquad (z \in G) \tag{4.1}$$

with some constants $m > 0$, $c_1 > 0$.

For an arbitrary complex number $a \in \mathbb{C}$ denote by $\nu^a_{Q_n}$ the measure that associates the mass $1/n$ with each of the a-values of the polynomial Q_n, that is, with the roots of the equation

$$Q_n(z) = a.$$

THEOREM 4. *Let L, $z_0 \in G := \operatorname{int} L$, α and c_0 be as in Theorem* 1, *and let h satisfy* (4.1). *Then for each complex number $a (\neq 0)$*

$$|(\mu_L - \nu^a_{Q_n})(A_{\sigma, \tau}(J))| \leqslant c \left(\frac{\log n}{n} \right)^{\alpha/(1+\alpha)} \qquad (n \geqslant 2) \tag{4.2}$$

for all subarcs J of L and all σ and τ such that

$$\sigma \geqslant \sigma_n := c_0 \left(\frac{\log n}{n} \right)^{\alpha/(1+\alpha)}, \qquad 1 > \tau \geqslant \tau_n := \sigma_n^{1/\alpha},$$

where the constant c is independent of J and n.

The exceptional role of the value $a = 0$ in Theorem 4 becomes obvious if we consider the system

$$\left\{ \left(\frac{n+1}{\pi} \right)^{1/2} z^n \right\}_{n \in \mathbb{N}_0}$$

of polynomials orthogonal on the unit disk D with respect to the weight function $h(z) \equiv 1$.

The example constructed below shows also that the restriction (4.1) in Theorem 4, in general, cannot be omitted.

In fact, let $G := D$,

$$h(z) = h(|z|) := \exp\left\{ -\exp\left\{ \frac{1}{1 - |z|} \right\} \right\} \qquad (z \in D).$$

It is easy to see that the function h does not satisfy (4.1) and $Q_n(z)=\lambda_n z^n$, where

$$\lambda_n^{-2}=2\pi \int_0^1 h(r)\, r^{2n+1}\, dr.$$

Setting (for sufficiently large n) $r_n := 1-1/(\log n)$, we have

$$(2\pi\lambda_n^2)^{-1} \leqslant r_n^{2n+1} + \exp\left\{-\exp\left\{\frac{1}{1-r_n}\right\}\right\} \leqslant \exp\left\{-c_1 \frac{n}{\log n}\right\}.$$

It means that the roots $z_1, ..., z_n$ of the equation $Q_n(z)=1$ satisfy

$$|z_j| = \lambda_n^{-1/n} \leqslant 1 - \frac{c_2}{\log n} \qquad (j=1, ..., n).$$

Thus, in this case the conclusion of Theorem 4 is violated.

It is interesting to compare the statement of Theorem 4 with Picard's theorem which asserts that an analytic function assumes in an arbitrary neighborhood of its essential singularity all finite complex values with at most one possible exception.

It turnes out that imposing supplementary restrictions on the geometry of L we can derive also some information about the zeros of Q_n.

THEOREM 5. *Let L, G, z_0, α and c_0 be as in Theorem* 1, *and suppose that for some $k \in \mathbb{N}$ the conformal mapping φ satisfies the condition*

$$\|\varphi^{(k)}\|_G = \infty. \tag{4.3}$$

Then there exist an infinite sequence $\Lambda \subseteq \mathbb{N}$ and a positive constant c depending only on z_0, c_0 and L such that if $n \in \Lambda$, then the counting measure $\nu_{\tilde{Q}_n}$ for the zeros of the orthogonal polynomial $\tilde{Q}_n(z) = Q_n(h, z)$, where $h(z) \equiv 1$, satisfies the inequality

$$|(\mu_L - \nu_{\tilde{Q}_n})(A_{\sigma,\tau}(J))| \leqslant c \left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}$$

for any subarc J of L and all σ and τ with the properties

$$\sigma \geqslant \sigma_n := c_0 \left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}, \qquad 1 > \tau \geqslant \tau_n := \sigma_n^{1/\alpha}.$$

Eiermann and Stahl [14] considered convex domains having a polygonial boundary, especially N-gons G_N, $N=3, 4, ...,$ which have their vertices at the Nth roots of unity. They conjectured that

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n>m} Z_n} \cap \partial G = \{z_1, ..., z_N\}, \tag{4.5}$$

where Z_n, $n \in \mathbb{N}$, denote the sets of zeros of $\tilde{Q}_n$ and $z_1, ..., z_N$ are the vertices of $\partial G = \partial G_N$. That is, only the vertices of G attract zeros of $\tilde{Q}_n$ (as $n \to \infty$).

This conjecture is false for $N \geqslant 5$. Indeed, in the neighborhood of z_j, $j=1, ..., N$, the conformal mapping φ has the representation

$$\varphi(z) = h_j((z-z_j)^{N/(N-2)}),$$

where the h_j are analytic in a disk around the origin with sufficiently small radius. Therefore, for $N \geqslant 5$, φ'' has a singularity at z_j and by Theorem 5

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n>m} Z_n} \supset \partial G,$$

which contradicts (4.5).

5. APPLICATION TO ZEROS OF BIEBERBACH POLYNOMIALS

As before, let G be a quasidisk and let $z_0 \in G$ be an arbitrary fixed point.

Denote by f_0 the conformal mapping of G onto a disk $\{w\colon |w| < r_0\}$ with $f_0(z_0)=0$, $f_0'(z_0)=1$ ($r_0 = r_0(G, z_0)$ is called the conformal radius of G with respect to z_0). It is obvious that $f_0 = r_0 \varphi$.

It is well known that f_0 minimizes the integral

$$\int_G |f'(z)|^2 \, dm(z) \tag{5.1}$$

in the class of all functions f analytic in G and normalized by the conditions

$$f(z_0)=0, \qquad f'(z_0)=1.$$

The expression (5.1) is minimized in the class

$$\left\{ f = p_n \in \bigcup_{k=1}^{n} \mathbb{P}_k \colon p_n(z_0)=0, \, p_n'(z_0)=1 \right\}$$

by exactly one polynomial π_n. This polynomial is called the nth Bieberbach polynomial for the region G.

In [5] it was proved that

$$\lim_{n\to\infty} \|f_0 - \pi_n\|_{\bar{G}} = 0. \tag{5.2}$$

According to Rouche's theorem it means that for n large enough π_n does not have zeros in G except for z_0.

It was shown in [25] that the distributions of the zeros of π_n and π_n' are governed by the location of the singularities of the mapping function f_0.

We will be interested in the case where all boundary points of G are accumulation points of zeros of $\{\pi_n\}$ or $\{\pi_n'\}$. By [25] the boundary $L = \partial G$ attracts zeros of π_n and π_n' iff the function f_0 cannot be analytically extended to a neighborhood of $\bar{G}$. This fact explains the sense of the restriction (4.3) imposed in the theorem below.

THEOREM 6. *Let L, G, z_0, α and c_0 be as in Theorem* 1, *and suppose that for some $k \in \mathbb{N}$ the function φ satisfies condition* (4.3). *Then there exist an infinite sequence $\Lambda \subseteq \mathbb{N}$ and positive constants c_1, c_2 depending only on z_0, c_0 and L such that if $n \in \Lambda$, then*

$$|(\mu_L - \nu_{\pi_n'})(A_{\sigma,\tau}(J))| \leqslant c_1 \left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}, \tag{5.3}$$

$$|(\mu_L - \nu_{\pi_n})(\bar{\Omega} \cap A_{\sigma,\tau}(J)) \leqslant c_2 \left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}, \tag{5.4}$$

for any subarc J of L and all σ and τ with the properties

$$\sigma \geqslant \sigma_n := c_0 \left(\frac{\log n}{n}\right)^{\alpha/(1+\alpha)}, \qquad 1 > \tau \geqslant \tau_n := \sigma_n^{1/\alpha}.$$

6. SOME AUXILIARY FACTS FROM THE THEORY OF QUASICONFORMAL MAPPINGS

If G is an arbitrary K-quasidisk, it is known (see [2, Chapter IV]) that the conformal mappings Ψ and φ can be extended to K_1-quasiconformal homeomorphisms ($K_1 = K_1(K, z_0) \geqslant 1$) of the extended complex plane onto itself with ∞ as fixed point. We keep the previous notations for these extensions. Note that the inverse functions $\Psi := \Phi^{-1}$ and $\psi := \varphi^{-1}$ will be K_1-quasiconformal mappings, too.

The following result (see [3, Lemma 1]) is useful in the study of the metric properties of the mappings Φ, Ψ, φ, ψ.

Lemma 1. *Let* $w = F(\zeta)$ *be a* K*-quasiconformal mapping of the plane onto itself, with* $F(\infty) = \infty$, $\zeta_j \in \mathbb{C}$, $w_j := F(\zeta_j)$, $j = 1, 2, 3$, *and* $|w_1 - w_2| \leqslant c_1 |w_1 - w_3|$. *Then* $|\zeta_1 - \zeta_2| \leqslant c_2 |\zeta_1 - \zeta_3|$ *and, in addition,*

$$\left|\frac{\zeta_1 - \zeta_3}{\zeta_1 - \zeta_2}\right| \leqslant c_3 \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^K,$$

where $c_i = c_i(c_1, K)$, $i = 2, 3$.

Let E be a bounded Jordan domain and let J be a subarc of ∂E.

Denote by $\Gamma = \Gamma(E, J, z)$ the family of all locally rectifiable cross-cuts $\gamma \subset E$ of E separating in E the point $z \in E$ from J.

For the harmonic measure $\omega(z, E, J)$ of J at the point z with respect to E the following estimate can be derived from [20, pp. 319–320] (see also [19, p. 6])

$$\omega(z, E, J) \leqslant c_1 \exp\{-\pi m(\Gamma)\}, \tag{6.1}$$

where $m(\Gamma)$ is the module of the family Γ [2, 22].

We recall three well-known facts concerning the notion of the module of a family of curves.

We begin with the comparison principle. Let Γ' and Γ'' be two families of curves and arcs. If every $\gamma' \in \Gamma'$ contains a $\gamma'' \in \Gamma''$ then

$$m(\Gamma') \leqslant m(\Gamma''). \tag{6.2}$$

Further, the module of the family

$$\Gamma_0 = \Gamma_0(r_1, r_2, \theta_1, \theta_2) := \{\gamma_\theta := \{re^{i\theta}: r_1 < r < r_2\}: \theta_1 < \theta < \theta_2\},$$

where $0 < r_1 < r_2 < \infty$, $0 < \theta_2 - \theta_1 \leqslant 2\pi$, can be calculated exactly, namely,

$$m(\Gamma_0) = \frac{\theta_2 - \theta_1}{\log(r_2/r_1)}. \tag{6.3}$$

The last property of the module of a family of curves important for our considerations is its quasi-invariance. It means that for any K-quasiconformal mapping F of some domain $E \subset \overline{\mathbb{C}}$ and any family Γ of arcs and curves $\gamma \subset E$,

$$K^{-1} m(F(\Gamma)) \leqslant m(\Gamma) \leqslant K m(F(\Gamma)). \tag{6.4}$$

7. CONSTRUCTION AND PROPERTIES OF THE INTERMEDIATE FUNCTIONS

Let l be an arbitrary subarc of the K-quasiconformal curve L and let α, C be the constants in (2.1).

Denote by

$$\omega(z) := \omega(z, E_{\sigma,\tau}, l_\sigma^+ \cup l_\tau^-),$$

where $\sigma > 0$, $0 < \tau < 1$, the harmonic measure of $l_\sigma^+ \cup l_\tau^-$ at the point $z \in E_{\sigma,\tau}$ with respect to $E_{\sigma,\tau}$.

LEMMA 2. *Let* $z \in A_{\sigma,\tau}(L \backslash l)$, $0 < \sigma < 1$, $\tau := \sigma^{1/\alpha}$. *Then there exist constants* $c_j = c_j(K, \alpha, C)$, $j = 1, 2$, *such that*

$$\omega(z) \leqslant c_1 \exp\left\{-c_2 \frac{d}{\sigma}\right\}, \tag{7.1}$$

where $d := d(\Phi(z_L), \Phi(l))$ *is the distance between* $\Phi(z_L)$ *and* $\Phi(l)$.

Proof. A glance at the function estimated shows that we may suppose without loss of generality that $d \geqslant c_3 \sigma$, where the sufficiently large constant $c_3 > 1$ will be chosen later.

Let ζ_1 and ζ_2 be the endpoints of the arc l. Consider the quadrilateral $Q := A_{\sigma,\tau}(L \backslash l)$ whose sides are the arcs

$$\gamma_j := \{\zeta \in E_{\sigma,\tau} : \zeta_L = \zeta_j\} \qquad (j = 1, 2),$$

$$\gamma_3 := (L \backslash l)_\tau^-, \qquad \gamma_4 = (L \backslash l)_\sigma^+.$$

By the maximum principle,

$$\omega(z) \leqslant \sum_{j=1}^{2} \omega(z, Q, \gamma_j). \tag{7.2}$$

The arc

$$\gamma_5 := \{\zeta \in E_{\sigma,\tau} : \zeta_L = z_L\}$$

divides Q into two new quadrilaterals Q_1 and Q_2. For definiteness we assume that

$$\gamma_2 \subset \partial Q_j \qquad (j = 1, 2).$$

Denote by Γ_j, $j=1, 2$, the family of all locally rectifiable arcs $\gamma \subset Q_j$ separating in Q_j the sides γ_j and γ_5, i.e., arcs with one endpoint on γ_3 and the other one on γ_4.

Applying (6.1) and the comparison principle for moduli of families of curves (6.2), we find that

$$\omega(z, Q, \gamma_j) \preccurlyeq \exp\{-\pi m(\Gamma_j)\} \qquad (j=1, 2). \tag{7.3}$$

Thus, the problem of a suitable estimation of the quantity $m(\Gamma_j)$ from below has to be our next target.

We begin with the inequality

$$m(\Gamma_j) \geqslant K_1^{-1} m(\Gamma'_j), \tag{7.4}$$

where $\Gamma'_j := \Phi(\Gamma_j)$, which follows from the relation (6.4).

Now, set

$$s := \Phi(L \backslash l) = \{e^{i\theta}: \theta_1 < \theta < \theta_2\},$$

$$s_k := \Phi(\gamma_k) \qquad\qquad (k=1, ..., 5),$$

$$t_j := \bar{s}_3 \cap \bar{s}_j, \qquad\qquad w_j := \bar{s} \cap \bar{s}_j \qquad (j=1, 2, 5).$$

Lemma 1 applied to the quasiconformal mapping $\Phi \circ \psi$ implies that for $j=1, 2, 5$,

$$\operatorname{diam}(s_j \cap D) \leqslant c_4 \, |w_j - t_j| \leqslant c_5 (1 - |t_j|).$$

Furthermore, for any $t \in s_3$,

$$1 - |t| \leqslant c_6 \sigma.$$

Indeed, consider the point $\zeta := (\varphi \circ \Psi)(t)$ and the arc $\gamma := (\varphi \circ \Psi)(s)$. Since by our assumption $\operatorname{diam} s \geqslant \sigma$ we have

$$\text{length } \gamma \geqslant c_7 \sigma^{1/\alpha}.$$

We select the points ξ_1 and $\xi_2 \in \gamma$ so that

$$\arg \xi_1 = \arg \zeta, \qquad |\xi_1 - \xi_2| = \frac{1}{2c_7} \sigma^{1/\alpha}.$$

It is important to recall that the mapping $\varphi \circ \Psi$ is quasiconformal, since this implies via Lemma 1 that for $\tau_j := (\Phi \circ \psi)(\xi_j)$, $j = 1, 2$, we have

$$1 - |t| \leqslant |t - \tau_1| \leqslant c_8\, |\tau_1 - \tau_2| \leqslant c_9\, |\xi_1 - \xi_2|^{\alpha} = c_6 \sigma.$$

Consider the family of arcs

$$\Gamma_j'' := \{\gamma_\theta := \{re^{i\theta} \colon 1 - c_6\sigma < r < 1 + \sigma\} \colon \theta_1 + c_{10}\sigma \leqslant \theta \leqslant \theta_2 - c_{10}\sigma\},$$

where $c_{10} := (\pi/2)\, c_5 c_6$. It should be noted that Γ_j'' will be defined correctly if we chose $c_3 > 2c_{10}$.

Moreover, if $c_3 > 3c_{10}$, then according to (6.3) and the comparison principle (6.2), we get

$$m(\Gamma_j') \geqslant m(\Gamma_j'') = \frac{\theta_2 - \theta_1 - 2c_{10}\sigma}{\log \dfrac{1+\sigma}{1 - c_6\sigma}} \geqslant c_{11}\, \frac{d}{\sigma}. \tag{7.5}$$

Hence, in virtue of (7.2)–(7.5), (7.1) is satisfied. ∎

Corollary. *Writing the assertion of Lemma 2 for the arc $L \backslash l$ instead of l, we obtain the following inequality*

$$1 - \omega(z) \leqslant c_1 \exp\left\{ -\frac{c_2}{\sigma}\, d(\Phi(z), \Phi(L \backslash l)) \right\} \qquad (z \in A_{\sigma, \sigma^{1/\alpha}}(l)).$$

Now, for $w \in \mathbb{C} \backslash \partial D$, $0 < \sigma < 1$ and $\tau := \sigma^{1/\alpha}$ set

$$\tilde{\omega}(w) := \begin{cases} \omega[\Psi(w)] & \text{if} \quad 1 < |w| \leqslant 1 + \sigma \\ \omega[\psi(w)] & \text{if} \quad 1 - \tau \leqslant |w| < 1 \\ 0 & \text{if} \quad |w| > 1 + \sigma \quad \text{or} \quad |w| < 1 - \tau. \end{cases}$$

We average this function in D and Δ separately in the following way. Let $K(z)$, $z \in \mathbb{C}$, be an arbitrary averaging kernel, i.e., $K(z)$ has in $\mathbb{C}$ partial derivatives of all orders,

$$K(z) = K(|z|) \geqslant 0 \qquad (z \in \mathbb{C}),$$

$$K(z) = 0 \qquad (|z| \geqslant 1),$$

$$\int_{\mathbb{C}} K(z)\, dm(z) = 1.$$

Consider the function

$$\tilde{g}(w) := \begin{cases} \dfrac{16}{\sigma^2} \displaystyle\int_{\mathbb{C}} \tilde{\omega}(t)\, K\left(\frac{4(t-w)}{\sigma}\right) dm(t) & \text{if} \quad 1+\dfrac{1}{2}\sigma \leqslant |w| \leqslant 1+\dfrac{3}{2}\sigma \\ \dfrac{16}{\tau^2} \displaystyle\int_{\mathbb{C}} \tilde{\omega}(t)\, K\left(\frac{4(t-w)}{\tau}\right) dm(t) & \text{if} \quad 1-\dfrac{3}{2}\tau \leqslant |w| \leqslant 1-\dfrac{1}{2}\tau \\ \tilde{\omega}(w) & \text{elsewhere in} \quad \mathbb{C}\backslash\partial D. \end{cases}$$

Note that the function $\tilde{g}$ has in $\mathbb{C}\backslash\partial D$ partial derivatives of all orders and satisfies the inequalities

$$0 \leqslant \tilde{g}(w) \leqslant 1 \qquad (w \in \mathbb{C}\backslash\partial D), \tag{7.6}$$

$$|\Delta\tilde{g}(w)| \leqslant \begin{cases} \sigma^{-2} & \text{if} \quad |w|>1 \\ \tau^{-2} & \text{if} \quad |w|<1. \end{cases} \tag{7.7}$$

Therefore, the function

$$g(z) := \begin{cases} \tilde{g}[\Phi(z)] & \text{if} \quad z \in \operatorname{ext} L \\ \tilde{g}[\varphi(z)] & \text{if} \quad z \in \operatorname{int} L \\ \omega(z) & \text{if} \quad z \in L \end{cases}$$

has in $\mathbb{C}$ partial derivatives of all orders.

Moreover, by the Green formula

$$\int_{\mathbb{C}} \Delta g(z)\, dm(z) = 0. \tag{7.8}$$

Next, applying the technique of [10], we can establish the inequality

$$\left| \int g(d\nu - d\mu) \right| \leqslant \frac{\delta_p}{\tau}. \tag{7.9}$$

Indeed, setting

$$\tilde{U}^{\mu-\nu}(w) := \begin{cases} U^{\mu-\nu}[\Psi(w)] & \text{if} \quad |w|>1 \\ U^{\mu-\nu}[\psi(w)] & \text{if} \quad |w|<1 \end{cases}$$

and using the representation of the function g by the Green formula

$$g(z) = \frac{1}{2\pi} \int_{\mathbb{C}} \Delta g(\zeta) \log|z-\zeta|\, dm(\zeta) \qquad (z \in \mathbb{C}),$$

we obtain according to (7.7) and (7.8)

$$\left|\int g(d\nu-d\mu)\right|=\frac{1}{2\pi}\left|\int_{\mathbb{C}} U^{\mu-\nu}(\zeta)\,\Delta g(\zeta)\,dm(\zeta)\right|$$

$$=\frac{1}{2\pi}\left|\int_{\mathbb{C}} (\varepsilon_p-U^{\mu-\nu}(\zeta))\,\Delta g(\zeta)\,dm(\zeta)\right|$$

$$\leqslant\frac{1}{2\pi}\int_{\Delta\cup D} (\varepsilon_p-\tilde{U}^{\mu-\nu}(w))\,|\Delta\tilde{g}(w)|\,dm(w)$$

$$\leqslant\sigma^{-2}\int_{1+\sigma/2}^{1+2\sigma} r\,\frac{1}{2\pi}\int_0^{2\pi} (\varepsilon_p-\tilde{U}^{\mu-\nu}(re^{i\theta}))\,d\theta\,dr$$

$$+\tau^{-2}\int_{1-2\tau}^{1-\tau/2} r\,\frac{1}{2\pi}\int_0^{2\pi} (\varepsilon_p-\tilde{U}^{\mu-\nu}(re^{i\theta}))\,d\theta\,dr$$

$$\leqslant\frac{\varepsilon_p}{\sigma}+\frac{\varepsilon_p-\tilde{U}^{\mu-\nu}(0)}{\tau}\leqslant\frac{\delta_p}{\tau}.$$

Further, set $\sigma_1:=\sigma/2$, $\tau_1:=\tau/2$. By the Green formula we have for $z\in L$

$$g(z)=\frac{1}{2\pi}\int_{L^+_{\sigma_1}\cup L^-_{\tau_1}}\left(\omega(\zeta)\,\frac{\partial}{\partial n}\log|\zeta-z|-\frac{\partial}{\partial n}\,\omega(\zeta)\log|\zeta-z|\right)|d\zeta|.$$

Integrating the last relation we get

$$\int g\,d\mu=-\frac{1}{2\pi}\int_{L^+_{\sigma_1}\cup L^-_{\tau_1}}\left(\omega(\zeta)\,\frac{\partial}{\partial n}\,U^{\mu}(\zeta)-\frac{\partial}{\partial n}\,\omega(\zeta)\,U^{\mu}(\zeta)\right)|d\zeta|$$

$$=\frac{1}{2\pi}\int_{L^+_{\sigma_1}}\left(\omega(\zeta)\,\frac{\partial}{\partial n}\log|\Phi(\zeta)|-\frac{\partial}{\partial n}\,\omega(\zeta)\log|\Phi(\zeta)|\right)|d\xi|. \qquad (7.10)$$

For $w\in\Delta$, set $\tilde{U}^{\mu}(w):=U^{\mu}(\Psi(w))$. Next, we analize the integrals on the right-hand side of (7.10). Note that

$$\frac{1}{2\pi}\int_{L^+_{\sigma_1}}\left(\omega(\zeta)\,\frac{\partial}{\partial n}\log|\Phi(\zeta)|\right)|d\zeta|$$

$$=\frac{1}{2\pi}\int_{|w|=1+\sigma_1}\frac{\tilde{\omega}(w)}{|w|}\,|dw|$$

$$=\mu(l)+\frac{1}{2\pi(1+\sigma_1)}\int_{|w|=1+\sigma_1}(\tilde{\omega}(w)-\chi(w))\,|dw|,$$

where

$$\chi(w) := \begin{cases} 1 & \text{if } \dfrac{w}{1+\sigma_1} \in \Phi(l) \\ 0 & \text{elsewhere in } \mathbb{C}. \end{cases}$$

Since by Lemma 2 and its Corollary

$$\frac{1}{2\pi(1+\sigma_1)} \int_{|w|=1+\sigma_1} (\tilde{\omega}(w) - \chi(w))\, |dw| \preccurlyeq \int_0^{2\pi} e^{-c_1 x/\sigma}\, dx \preccurlyeq \sigma,$$

we have

$$\left| \frac{1}{2\pi} \int_{L^+_{\sigma_1}} \omega(\zeta) \frac{\partial}{\partial n} \log |\Phi(\zeta)|\, |d\zeta| - \mu(l) \right| \preccurlyeq \sigma. \tag{7.11}$$

The same reasoning can be applied to the second integral. We only need to add the following simple consequence of Schwarz's formula: For w with $|w| = 1 + \sigma_1$,

$$\begin{aligned} |\text{grad}\, \tilde{\omega}(w)| &\leqslant \frac{1}{\pi} \int_{|\tau - w| = \sigma_1} \frac{|\tilde{\omega}(\tau) - \tilde{\omega}(w)|}{|\tau - w|^2}\, |d\tau| \\ &\preccurlyeq \frac{1}{\sigma} \exp\left\{ -c_2 \frac{d(w/(1+\sigma_1), \{w_1, w_2\})}{\sigma} \right\}, \end{aligned}$$

where w_1 and w_2 are the endpoints of $\Phi(l)$, and therefore

$$\left| \int_{|w|=1+\sigma_1} \frac{\partial}{\partial n} \tilde{\omega}(w)\, |dw| \right| \preccurlyeq \frac{1}{\sigma} \int_0^{2\pi} e^{-c_2 x/\sigma}\, dx \preccurlyeq 1.$$

Hence

$$\begin{aligned} &\left| \frac{1}{2\pi} \int_{L^+_{\sigma_1}} \frac{\partial}{\partial n} \omega(\zeta) \log |\Phi(\zeta)|\, |d\xi| \right| \\ &\qquad = \frac{1}{2\pi} \log(1+\sigma_1) \left| \int_{|w|=1+\sigma_1} \frac{\partial}{\partial n} \tilde{\omega}(w)\, |dw| \right| \preccurlyeq \sigma. \end{aligned} \tag{7.12}$$

Combining (7.10)–(7.12) we get

$$\left| \int g\, d\mu - \mu(l) \right| \preccurlyeq \sigma. \tag{7.13}$$

Apart from this we have in view of (7.9) and (7.13)

$$\left|\int g\, dv - \mu(l)\right| \preccurlyeq \left(\sigma + \frac{\delta_p}{\tau}\right). \tag{7.14}$$

8. PROOF OF THEOREM 1

Without loss of generality we may assume that δ_p and, consequently, σ_n are sufficiently small.

By virtue of [10, Remarks 1 and 2], we have

$$\nu(\text{ext } L_t^+) \leqslant \frac{\varepsilon_p}{\log(1+t)} \qquad (t>0),$$

$$\nu(\text{int } L_t^-) \leqslant \frac{\varepsilon_p + U^{\nu-\mu}(z_0)}{\log(1/(1-t))} \qquad (0<t<1).$$

Therefore, for $0<\sigma<1/2$ and $0<\tau<1/2$,

$$|(\mu-\nu)(E_{\sigma,\tau})| = \nu(\mathbb{C}\setminus E_{\sigma,\tau}) \preccurlyeq \left(\frac{\varepsilon_p}{\sigma} - \frac{\varepsilon_p + U^{\nu-\mu}(z_0)}{\tau}\right). \tag{8.1}$$

A routine category argument shows that in order to prove inequality (2.3), it is sufficient to estimate the quantity $(\nu-\mu)(A_{\sigma,\tau}(J))$ only for $\sigma=\sigma_n$ and $\tau=\tau_n$.

Now, for $t>0$ set

$$\gamma := \Phi(J) = \{e^{i\theta}: \theta_1 \leqslant \theta \leqslant \theta_2\},$$

$$\gamma_t := \{e^{i\theta}: \theta_1 - t\sigma_n \leqslant \theta \leqslant \theta_2 + t\sigma_n\},$$

$$J_t := \Psi(\gamma_t).$$

First, we consider the case $\mu(J) \leqslant \sigma_n$. Let M be a sufficiently large constant to be chosen later. For $l := J_M$ and $\sigma := 2\sigma_n$, let g be the function constructed in the previous section.

By the Corollary to Lemma 2 we can choose M such that

$$g(z) = \omega(z) \geqslant \tfrac{1}{2} \qquad (z \in A_{\sigma_n, \tau_n}(J)).$$

Therefore, by (7.14)

$$\tfrac{1}{2}\, \nu(A_{\sigma_n,\tau_n}(J)) \leqslant \int g\, d\nu \preccurlyeq \sigma_n. \tag{8.2}$$

Now, let $\sigma_n < \mu(J) =: a$. Without loss of generality we assume that

$$\gamma = \Phi(J) = \{e^{i\theta}\colon -a\pi \leqslant \theta \leqslant a\pi\}.$$

For $l := J$ and $\sigma := \sigma_n$ consider the function g from above. By virtue of (7.14) we obtain

$$\int g\, dv \geqslant a - c_1 \sigma_n. \tag{8.3}$$

To estimate the left-hand term in (8.3) we construct the following division of the circular arc$\{|w| = 1\} \setminus \gamma$.

We distinguish two cases.

If $1 - a \leqslant \sigma_n$, then

$$\Gamma_1 := \{e^{i\theta}\colon a\pi \leqslant \theta \leqslant \pi\}.$$

If $1 - a > \sigma_n$, then

$$\Gamma_k := \{e^{i\theta}\colon (a + k\sigma_n)\, \pi \leqslant \theta \leqslant (a + (k+1)\, \sigma_n)\, \pi\}$$
$$\left(k = 0, ..., k_0 := \left[\frac{1-a}{\sigma_n}\right] - 1\right),$$
$$\Gamma_{k_0+1} := \{e^{i\theta}\colon (a + (k_0+1)\, \sigma_n)\, \pi \leqslant \theta \leqslant \pi\}.$$

In both cases let

$$\Gamma_{-k} := \{e^{i\theta}\colon e^{-i\theta} \in \Gamma_k\},$$
$$\Gamma'_{\pm k} := \Psi(\Gamma_{\pm k}).$$

Since by (8.2),

$$v(A_{\sigma_n, \tau_n}(\Gamma'_{\pm k})) \lessapprox \sigma_n,$$

we have owing to Lemma 2, its Corollary, (7.6) and (8.1)

$$\begin{aligned}\int g\, dv &\leqslant v(\mathbb{C} \setminus E_{\sigma_n, \tau_n}) + v(A_{\sigma_n, \tau_n}(J)) \\ &\quad + c_2 \sum_{k=0}^{k_0} e^{-c_3 k} [v(A_{\sigma_n, \tau_n}(\Gamma'_{k+1})) + v(A_{\sigma_n, \tau_n}(\Gamma'_{-(k+1)}))] \\ &\leqslant v(A_{\sigma_n, \tau_n}(J)) + c_4 \sigma_n.\end{aligned}$$

Comparing the last inequality with (8.3), we have for any subarc $J \subset L$

$$(\nu - \mu)(A_{\sigma_n, \tau_n}(J)) \geqslant -c_5 \sigma_n. \tag{8.4}$$

In order to get the upper estimate of $(\nu - \mu)(A_{\sigma_n, \tau_n}(J))$ we need, due to the relation

$$\begin{aligned}(\nu - \mu)(A_{\sigma_n, \tau_n}(J)) = -(\nu - \mu)(A_{\sigma_n, \tau_n}(L \backslash J)) - \nu(\mathbb{C} \backslash E_{\sigma_n, \tau_n}) \\ + \nu((A_{\sigma_n, \tau_n}(J) \cap A_{\sigma_n, \tau_n}(L \backslash J)),\end{aligned}$$

merely to apply (8.2) and (8.4) to the arc $L \backslash J$.

9. PROOF OF THEOREM 2

Notice that the proof of Theorem 2 follows the same ideas as the proof of Theorem 1. We give only a sketch of the reasoning to show how the scheme has to be modified in order to obtain the result.

Let z_1 and z_2 be the endpoints of the quasiconformal arc L. Let l be a subarc of L. Consider the set

$$l_\sigma := \{\zeta \in L_\sigma : \zeta_L \in L\} \qquad (\sigma > 0)$$

and the function

$$\omega(z) := \omega(z, E_\sigma, l_\sigma) \qquad (\sigma > 0),$$

that is, the harmonic measure of l_σ at z with respect to E_σ.

First, we establish the analogue of Lemma 2.

Lemma 3. *Let one of the endpoints of l coincide with z_1. Denote by $z_3 \in L$ the other endpoint of l. There exist constants $c_j = c_j(K, \alpha, C)$, $j = 1, 2$, such that for $z \in A_\sigma(L \backslash l)$ the relation*

$$\omega(z) \leqslant c_1 \exp\left\{-c_2 \frac{\mu(L(z_3, z_L))}{\sigma^\alpha}\right\} \tag{9.1}$$

holds, where $C > 0, 0 < \alpha \leqslant 1$ are the constants from inequality (2.4), *and $L(\xi, \zeta)$ denotes the subarc of L joining the points ξ and $\zeta \in L$.*

Proof. Set $S := L(z_3, z_L)$. We may assume that $\mu(S) \geqslant c_3 \sigma^\alpha$, where the constant $c_3 > 1$ is large enough.

Divide S by points $\zeta_0 := z_L, \zeta_1, ..., \zeta_k := z_3$ into subarcs $S_i := L(\zeta_i, \zeta_{i+1})$, $i = 0, ..., k-1$, such that

$$\sigma/2 \leqslant \min_{j=1,2} \operatorname{diam} \Phi_j(S_i) \leqslant \sigma \qquad (i = 1, ..., k-1).$$

The possibility to perform this procedure for sufficiently large c_3 follows from (2.4).

Moreover, according to (2.4)

$$\max_{j=1,2} \operatorname{diam} \Phi_j(S_i) \preccurlyeq \sigma^\alpha.$$

Therefore

$$k \succcurlyeq \mu(S)\, \sigma^{-\alpha}.$$

We consider the arc

$$\gamma_\sigma(\zeta) := \{\xi \in E_\sigma : \xi_L = \zeta\}.$$

Reasoning exactly as in the proof of Theorem 1 we can show that

$$m(\{\gamma_\sigma(\zeta) : \zeta \in S_i\}) \succcurlyeq 1.$$

Recalling the composition law [2], we have

$$m(\{\gamma_\sigma(\zeta) : \zeta \in S\}) \geqslant \sum_{i=0}^{k-1} m(\{\gamma_\sigma(\zeta) : \zeta \in S_i\}) \succcurlyeq k \succcurlyeq \frac{\mu(S)}{\sigma^\alpha}.$$

Finally, applying the comparison principle (6.2) and relation (6.1) we obtain the desired estimate (9.1). ∎

Observe that the inequality

$$1 - \omega(z) \leqslant c_1 \exp\left\{-c_2 \frac{\mu(L(z_3, z_L))}{\sigma^\alpha}\right\} \qquad (z \in A_\sigma(l)) \tag{9.2}$$

is an immediate consequence of relation (9.1) written for $L \backslash l$.

If $l = L(\zeta_1, \zeta_2)$, $\zeta_1, \zeta_2 \in L$, is an arbitrary subarc of L we can express the function $\omega(z)$ as a difference of two functions of the shape as in Lemma 3 (corresponding to the arcs $L(z_1, \zeta_2)$ and $L(z_1, \zeta_1)$). Therefore, in this case we can write the appropriate analogues of (9.1) and (9.2) describing the behaviour of $\omega(z)$.

Now, consider the following functions

$$\tilde{\omega}(w) := \begin{cases} \omega[\Psi(w)] & \text{if} \quad 1 < |w| \leqslant 1+\sigma \\ 0 & \text{if} \quad |w| > 1+\sigma, \end{cases}$$

$$\tilde{g}(w) := \begin{cases} \dfrac{16}{\sigma^2} \displaystyle\int_{\mathbb{C}} \tilde{\omega}(t)\, K\left(\dfrac{4(t-w)}{\sigma}\right) dm(t) & \text{if} \quad 1+\dfrac{1}{2}\sigma \leqslant |w| \leqslant 1+\dfrac{3}{2}\sigma \\ \tilde{\omega}(w) & \text{elsewhere in } \Delta, \end{cases}$$

where $K(z)$ is an averaging kernel,

$$g(z) := \begin{cases} \tilde{g}[\Phi(z)] & \text{if} \quad z \in \Omega \\ \omega(z) & \text{if} \quad z \in L. \end{cases}$$

For the function g the analogue of inequality (7.9) takes the form

$$\left| \int g(dv - d\mu) \right| \leqslant \frac{\varepsilon_p}{\sigma},$$

and the analogue of inequality (7.13) has the form

$$\left| \int g\, d\mu - \mu(l) \right| \leqslant \sigma^{\alpha}.$$

As before, the case of interest is when $\sigma = \sigma_n$ is small enough. To see that $(v-\mu)(A_{\sigma_n}(J))$ is appropriately bounded, we have only to repeat practically word by word the last reasoning in the proof of Theorem 1.

10. PROOF OF THEOREM 4

In order to establish (4.2) we are going to apply Theorem 1 to the monic polynomial

$$p(z) = p_n(z) := \frac{Q_n(z) - a}{\lambda_n}. \tag{10.1}$$

Therefore, the appropriate estimates of the quantity $\|Q_n\|_{\bar{G}}$ and the leading coefficient λ_n will be the target of our next investigation.

We begin with the following assertion.

LEMMA 4. *For each point $z_0 \in G$,*

$$\lim_{n\to\infty} Q_n(z_0) = 0. \tag{10.2}$$

Proof. As before, let $\psi(w) = \psi_{z_0}(w)$ be the conformal mapping of the unit disk D onto G with $\psi(0) = z_0$, $\psi'(0) > 0$.

The mapping ψ transforms the polynomial

$$K_n(z) := \sum_{j=0}^{n} \overline{Q_j(z_0)}\, Q_j(z)$$

to a function $\widetilde{K}_n(w) := K_n[\psi(w)]$, and for $w \in D$,

$$\widetilde{K}_n(w)\, \psi'(w) = \sum_{k=0}^{\infty} a_k w^k,$$

where $a_0 = K_n(z_0)\, \psi'(0)$.

If we take into account that

$$d(\psi(w), L) \succcurlyeq (1 - |w|)^2 \qquad (w \in D),$$

(see, for example, [7, p. 61]), we get

$$\begin{aligned} K_n(z_0) &= \sum_{j=0}^{n} |Q_j(z_0)|^2 \\ &= \int_G h(z)\, |K_n(z)|^2\, dm(z) \\ &\succcurlyeq \int_D (1 - |w|)^{2m} \left| \sum_{k=0}^{\infty} a_k w \right|^2 dm(w) \\ &\geqslant 2\pi \sum_{k=0}^{\infty} |a_k|^2 \int_0^{1/2} (1 - r)^{2m}\, r^{2k+1}\, dr \\ &\succcurlyeq |a_0|^2 = (K_n(z_0)\, \psi'(0))^2. \end{aligned}$$

Since

$$K_n(z_0) \preccurlyeq [\psi'(0)]^{-2} \preccurlyeq 1, \tag{10.3}$$

it follows that the series

$$\sum_{j=0}^{\infty} |Q_j(z_0)|^2$$

converges, from which (10.2) immediately follows. ∎

Further, we recall one auxiliary result which is implicitly given in [1].

Lemma 5. *Let $E \subset \mathbb{C}$ be an arbitrary bounded Jordan domain, $J := \partial E$. Next, let J_t^+, $t > 0$, be the $(1+t)$-level curve of the conformal mapping $\Pi: \overline{\mathbb{C}} \setminus \overline{E} \to \Delta$ with $\Pi(\infty) = \infty$, $\Pi'(\infty) > 0$. Then for each monic polynomial $p(z) = p_n(z) = z^n + \cdots$, $n \in \mathbb{N}$, and $t > 0$*

$$\int_{\text{int } J_t^+)\setminus \overline{E}} |p(z)|^2 \, dm(z) \geqslant 2\pi (\text{cap } J)^{2(n+1)} \, t. \tag{10.4}$$

Proof. For an arbitrary fixed point $z_0 \in E$ set

$$q(z) = q_{n,z_0}(z) := \int_{z_0}^{z} p(\zeta) \, d\zeta,$$

$$\tilde{q}(w) := q[\Pi^{-1}(w)] \qquad (w \in \Delta),$$

$$I_t := \int_{\text{int } J_t^+} |p(z)|^2 \, dm(z).$$

Then by the analytic Green formula, we may write

$$I_t = \frac{1}{2i} \int_{J_t^+} p(z) \, \overline{q(z)} \, dz = \frac{1}{2i} \int_{|w| = 1+t} \tilde{q}'(w) \, \overline{\tilde{q}(w)} | \, dw.$$

Using the Laurent series expansion of the function $\tilde{q}$ in a neighbourhood of ∞, i.e.,

$$\tilde{q}(w) = \frac{c^{n+1}}{n+1} w^{n+1} + \sum_{k=0}^{n} b_k w^k + \sum_{k=1}^{\infty} \frac{c_k}{w^k},$$

where $c := \text{cap } J$, we find that

$$I_t = \pi \left(\frac{c^{2(n+1)}}{n+1} (1+t)^{2(n+1)} + \sum_{k=1}^{n} k \, |b_k|^2 \, (1+t)^{2k} - \sum_{k=1}^{\infty} k \, |c_k|^2 \, (1+t)^{-2k} \right).$$

Hence,

$$\begin{aligned} \int_{(\text{int } J_t^+)\setminus \tilde{E}} |p(z)|^2 \, dm(z) &= I_t - \lim_{t \to 0} I_t \\ &\geqslant \pi \frac{c^{2(n+1)}}{n+1} \left[(1+t)^{2(n+1)} - 1 \right] > 2\pi c^{2(n+1)} t, \end{aligned}$$

which is the assertion in (10.4). ∎

LEMMA 6. *Let L be quasiconformal. Then the double inequality*

$$c_1 n^{-1/2} \leqslant \lambda_n (\operatorname{cap} L)^n \leqslant c_2 n^c \qquad (n \in \mathbb{N}) \tag{10.5}$$

holds with some positive constants c, c_1, c_2 independent of n.

Proof. Let $\Phi_n(z) = \rho^{-n} z^n + \cdots$, $\rho := \operatorname{cap} L$, be the nth Faber polynomial for the closed domain $\bar{G}$. An argument of [21] shows that

$$\|\Phi_n\|_{\bar{G}} \preccurlyeq n^{1/2}.$$

At the same time, using the expansion

$$\Phi_n = (\rho^n \lambda_n)^{-1} Q_n + d_{n-1} Q_{n-1} + \cdots + d_0 Q_0,$$

we obtain

$$\int_G h(z) \, |\Phi_n(z)|^2 \, dm(z) = (\rho^n \lambda_n)^{-2} + \sum_{k=0}^{n-1} |d_k|^2 \geqslant (\rho^n \lambda_n)^{-2},$$

from which the left-hand side of (10.5) simply follows.

It should be pointed out that we did not use the assumption of quasiconformality of L.

In order to establish the right-hand side of (10.5) we set

$$L_{-u} := \{\zeta \in G: |\Phi(\zeta)| = 1 - u\} \qquad (0 < u \leqslant 1)$$

and denote by Φ_{-u} the conformal mapping of $\operatorname{ext} L_{-u}$ onto Δ normalized by $\Phi_{-u}(\infty) = \infty$, $\Phi'_{-u}(\infty) > 0$. For $\zeta \in L$ [6, Lemma 3] implies that

$$c_3 u \leqslant |\Phi_{-u}(\zeta)| - 1 \leqslant c_4 u, \tag{10.6}$$

which yields

$$\operatorname{cap} L_{-u} \leqslant \rho \leqslant (1 + c_4 u) \operatorname{cap} L_{-u}. \tag{10.7}$$

We observe that if L is K-quasiconformal, $K \geqslant 1$, then L_{-u} is K^2-quasiconformal. Consequently, the mapping Φ_{-u} can be extended to a K^4-quasiconformal mapping of $\bar{\mathbb{C}}$ onto itself, a fact that makes it possible to use Lemma 1.

Hence, for

$$\zeta \in M_u := \left\{\zeta \in G: 1 \leqslant |\Phi_{-u}(\zeta)| \leqslant 1 + \frac{c_3}{2} u\right\}$$

we have

$$d(\zeta, L) \succcurlyeq u^{K^4}.$$

Applying Lemma 5 with $u := 1/n$, $E := \operatorname{int} L_{-u}$ and $t := c_3 u/2$ we obtain the following relation:

$$\begin{aligned}\lambda_n^{-2} &= \int_G h(z) \left| \frac{Q_n(z)}{\lambda_n} \right|^2 dm(z) \\ &\succcurlyeq n^{-mK^4} \int_{M_{1/n}} \left| \frac{Q_n(z)}{\lambda_n} \right|^2 dm(z) \\ &\succcurlyeq n^{-mK^4-1} (\operatorname{cap} L_{-1/n})^{2(n+1)},\end{aligned}$$

which, in view of (10.7), yields the right-hand side of (10.5). ∎

We note that the construction of the curve $L_{-1/n}$ is closely related to the possibility of estimating from above the uniform norm of orthogonal polynomials along $\bar{G}$, which is needed for our proof.

In fact, since for each point $z \in L_{-1/n}$ and $d := d(z, L)/2$,

$$\begin{aligned}|Q_n(z)|^2 &\leqslant \frac{1}{2\pi d^2} \int_{|\zeta - z| \leqslant d} |Q_n(\zeta)|^2 \, dm(\zeta) \\ &\preccurlyeq d^{-(m+2)} \int_{|\xi - z| \leqslant d} h(\zeta)\, |Q_n(\zeta)|^2 \, dm(\zeta) \leqslant d^{-(m+2)} \preccurlyeq n^c,\end{aligned}$$

by the classical Bernstein–Walsh lemma and relation (10.6), we have

$$\| Q_n \|_{\bar{G}} \preccurlyeq n^{c/2}. \tag{10.8}$$

Finally, an elementary computation for the monic polynomial (10.1) shows that by virtue of (10.2), (10.5), and (10.8),

$$\delta_p \preccurlyeq \frac{\log n}{n} \qquad (n \geqslant 2).$$

Thus, the assertion of Theorem 4 is a direct consequence of Theorem 1.

11. PROOF OF THEOREMS 5 AND 6

We begin with the sequence of orthogonal polynomials $\tilde{Q}_n(z) := Q_n(h, z)$, $n \in \mathbb{N}_0$, for the case of the weight function $h(z) \equiv 1$.

Set

$$K(z)=K(z_0, z):=\sum_{j=0}^{\infty} \overline{\tilde{Q}_j(z_0)}\, \tilde{Q}_j(z)$$

It is well-known [17] that Bieberbach polynomials and their derivatives can be represented as follows

$$\pi_n(z)=\frac{1}{K_{n-1}(z_0)} \int_{z_0}^{z} K_{n-1}(\zeta)\, d\zeta,$$

$$\pi_n'(z)=\frac{K_{n-1}(z)}{K_{n-1}(z_0)},$$

where

$$K_n(z):=\sum_{j=0}^{n} \overline{\tilde{Q}_j(z_0)}\, \tilde{Q}_j(z).$$

Lemma 7. *Under the assumptions of Theorem* 5 *and* 6 *there exists a sufficiently large constant* $k>0$ *such that*

$$\limsup_{j\to\infty} |\tilde{Q}_j(z_0)|\, j^k>0. \tag{11.1}$$

Proof. According to (10.8) and the Markov type inequality (see, for example, [7, p. 187])

$$\|\tilde{Q}_n^{(m)}\|_{\bar{G}} \leqslant c_1 n^{2m}\, \|\tilde{Q}_n\|_{\bar{G}} \leqslant c_2 n^{c+2m},$$

where $c=c(G)$, $c_j=c_j(G, m)$, $j=1, 2$, are positive constants. Therefore, if we assume to the contrary that (11.1) is not true, i.e.,

$$\limsup_{j\to\infty} |\tilde{Q}_j(z_0)|\, j^k=0$$

for any $k>0$, then the function

$$r_0\varphi'(z)=f_0'(z)=\frac{K(z)}{K(z_0)}$$

and all its derivatives possess bounded uniform norm along G. That contradicts our assumption (4.3). ∎

According to Lemma 7 there exist constants $c>0$, $k\in\mathbb{N}$ and an infinite sequence $\Lambda\subseteq\mathbb{N}$ such that

$$|\tilde{Q}_n(z_0)|\geqslant cn^{-k}\qquad(n\in\Lambda).\tag{11.2}$$

Hence, for $n\in\Lambda$ and $p:=\tilde{Q}_n/\lambda_n$ we obtain according to Lemma 6, (10.8) and (11.2) that

$$\delta_p\preccurlyeq\frac{\log n}{n},$$

and (4.4) follows directly from Theorem 1.

Further, for $n\in\Lambda$ we write

$$\pi'_{n+1}(z)=\gamma_n z^n+\cdots,\qquad \gamma_n:=\overline{\frac{Q_n(z_0)}{K_n(z_0)}}\,\lambda_n.$$

By virtue of (10.2), (10.3), (10.5), and (11.2)

$$|\gamma_n|\preccurlyeq\lambda_n\preccurlyeq n^c(\operatorname{cap}L)^{-n},$$
$$|\gamma_n|\succcurlyeq n^{-(1/2+k)}(\operatorname{cap}L)^{-n}.$$

Now, consider the monic polynomial

$$p(z)=p_n(z):=\frac{\pi'_{n+1}(z)}{\gamma_n}\qquad(n\in\Lambda).\tag{11.3}$$

For its uniform norm along $\bar{G}$ we have by the already mentioned Markov type inequality [7, p. 187] and (5.2)

$$\|p\|_{\bar{G}}\preccurlyeq n^2\,\|\pi_{n+1}\|_{\bar{G}}\,n^{1/2+k}(\operatorname{cap}L)^n\preccurlyeq n^{3+k}(\operatorname{cap}L)^n.$$

In addition

$$p(z_0)=\frac{1}{\gamma_n}\succcurlyeq n^{-c}(\operatorname{cap}L)^n.$$

Applying Theorem 1 to the polynomial p given by (11.3) we get (5.3).

For the proof of (5.4) we note that $\|\pi_n\|_{\bar{G}}\preccurlyeq 1$, $|\pi_n(z_0')|\succcurlyeq 1$ for some fixed point $z_0'\in G$, $z_0'\neq z_0$, and the leading coefficient of π_n differs from the corresponding one of π_n' only by a factor $1/n$, which does not effect the estimates needed for applying Theorem 1. Thus, by the same reasoning as above, applying Theorem 1 with some fixed point $z_0'\in G$, $z_0'\neq z_0$ instead of z_0, we get (5.4).

12. ONE REMARK

In the formulation of the main Theorems 1 and 2 we have used estimates from above for the potential $U^{\mu-\nu}(z)$ along $\mathbb{C}$. However, in the proof of these results we needed estimates of $U^{\mu-\nu}(z)$ only on some subsets of $\mathbb{C}$, namely on

$$\operatorname{ext} L^{+}_{\sigma_n/2} \cup \operatorname{int} L^{-}_{\sigma_n^{1/\alpha}/2}$$

in Theorem 1 and

$$\operatorname{ext} L_{\sigma_n/2}$$

in Theorem 2.

Moreover, if $U^{\mu-\nu}(z)$ is harmonic in $\mathbb{C}\backslash L$, which corresponds to the fact that all zeros of p belong to L, a lower bound of $U^{\mu-\nu}(z)$ can be used in the proof of the basic inequality (7.9) as well.

We display this approach writing the analogue of Theorem 2 only (for details, see [8]).

THEOREM 7. *Let L be a quasiconformal arc satisfying condition* (2.4). *Suppose that p is a monic polynomial such that all its zeros belong to L. Then there exists a constant $c>0$ depending only on L, α, C such that*

$$|(\mu_L-\nu_p)(J)| \leqslant c\left(\frac{\varepsilon^*(\sigma)}{\sigma}+\sigma^\alpha\right) \qquad (0<\sigma<1)$$

for all subarcs J of L, where

$$\varepsilon^*(\sigma) := -\inf_{z\in L_\sigma} U^{\mu_L-\nu_p}(z) \qquad (\sigma>0).$$

If all zeros $z_1, ..., z_n$ of a monic polynomial $p\in\mathbb{P}_n$, $n\geqslant 2$, are at the same time simple and for some $n\leqslant A_n\leqslant e^{n/e}$ satisfy

$$|p'(z_j)| \geqslant \frac{1}{A_n}(\operatorname{cap} L)^n \qquad (j=1, ..., n),$$

there is a simple way to show that

$$\varepsilon^*(\sigma) \leqslant \varepsilon^*\left(\frac{1}{n}\right) \leqslant c_1 \frac{\log A_n}{n} \qquad \left(\frac{1}{n}\leqslant\sigma<1\right)$$

(see, for example, [11]). Thus, taking in Theorem 7

$$\sigma = \left(\frac{\log A_n}{n}\right)^{1/(1+\alpha)},$$

we get for any subarc J of L

$$|(\mu_L - \nu_p)(J)| \leqslant c_2 \left(\frac{\log A_n}{n}\right)^{\alpha/(1+\alpha)}. \tag{12.1}$$

An example showing for $L = [-1, 1]$ the sharpness of (12.1) is constructed in [8].

REFERENCES

1. F. G. Abdulaev, "On orthogonal polynomials in domains with quasiconformal boundary," dissertation, University of Donetsk, 1986. [In Russian]
2. L. V. Ahlfors, "Lectures on Quasiconformal Mappings," Van Nostrand, Princeton, NJ, 1966.
3. V. V. Andrievskii, Some properties of continua with a piecewise quasiconformal boundary, *Ukrain. Mat. Zh.* **32** (1980), 435–440. [In Russian]
4. V. V. Andrievskii, Direct theorems of approximation theory on quasiconformal arcs, *Math. USSR Izv.* **16** (1981), 221–238.
5. V. V. Andrievskii, Convergence of Bieberbach polynomials in domains with quasiconformal boundary, *Ukrain. Mat. Zh.* **32** (1983), 273–277. [In Russian]
6. V. V. Andrievskii, A constructive characterization of harmonic functions in domains with quasiconformal boundary, *Math. USSR Izv.* **34** (1990), 441–454.
7. V. V. Andrievskii, V. I. Belyi, and V. K. Dzjadyk, "Conformal Invariants in Constructive Theory of Functions of Complex Variable," World Federation, Atlanta, GA, 1995.
8. V. V. Andrievskii and H.-P. Blatt, Erdős–Turán-type theorems on piecewise smooth curves and arcs, *J. Approx. Theory* **88** (1997), 109–134.
9. P. P. Belinskii, "General Properties of Quasiconformal Mappings," Nauka, Novosibirsk, 1974. [In Russian]
10. H.-P. Blatt and R. Grothmann, Erdős–Turán theorems on a system of Jordan curves and arcs, *Constr. Approx.* **7** (1991), 19–47.
11. H.-P. Blatt and H. Mhaskar, A general discrepancy theorem, *Ark. Mat.* **31** (1993), 219–246.
12. H.-P. Blatt, E. B. Saff, and M. Simkani, Jentzsch–Szegő type theorems for the zeros of best approximants, *J. London Math. Soc.* **38** (1988), 307–316.
13. H.-P. Blatt, E. B. Saff, and V. Totik, The distribution of extreme points in best complex polynomial approximation, *Constr. Approx.* **5** (1989), 357–370.
14. M. Eiermann and H. Stahl, Zeros of orthogonal polynomials on regular N-gons, *in* "Linear and Complex Analysis Problem Books" (V. P. Harin and N. K. Nikolski, Eds.), Vol. 2, Lecture Notes in Math. **1574** (1994), 187–189.
15. P. Erdős and P. Turán, On the uniformly-dense distribution of certain sequences of points, *Ann. of Math.* **41** (1940), 162–173.
16. P. Erdős and P. Turán, On the distribution of roots of polynomials, *Ann. of Math.* **51** (1950), 105–119.

17. D. Gaier, "Lectures on Complex Approximation," Birkhäuser, Boston, 1987.
18. R. Grothmann, "Interpolation points and zeros of polynomials in approximation theory," Habilitationsschrift, Katholische Universität Eichstätt, 1992.
19. K. Haliste, Estimates of harmonic measures, *Ark. Math.* **6** (1967), 1–31.
20. J. Hersch, Longuers extrémales et théorie des fonctions, *Comment. Math. Helv.* **29** (1955), 301–337.
21. O. Kövary and Ch. Pommerenke, On Faber polynomials and Faber expansions, *Math. Z.* **99** (1967), 193–206.
22. O. Lehto and K. I. Virtanen, "Quasiconformal Mappings in the Plane," 2nd ed., Springer-Verlag, New York, 1973.
23. F. D. Lesley, On interior and exterior conformal mappings of the disk, *J. London Math. Soc.* **20** (1979), 67–78.
24. F. D. Lesley, Conformal mappings of domains satisfying a wedge condition, *Proc. Amer. Math. Soc.* **93** (1985), 483–488.
25. N. Papamichael, E. B. Saff, and J. Gong, Asymptotic behavior of zeros of Bieberbach polynomials, *J. Comput. Appl. Math.* **34** (1991), 325–342.
26. Ch. Pommerenke, Polynome und konforme Abbildung, *Monatsh. Math.* **69** (1965), 58–61.
27. Ch. Pommerenke, "Boundary Behaviour of Conformal Maps," Springer-Verlag, New York, 1992.
28. S. Rickman, Characterization of quasiconformal arcs, *Ann. Acad. Sci. Fenn. Ser. AI Math.* **395** (1966), 1–30.
29. P. Sjögren, Estimates of mass distributions from their potentials and energies, *Ark. Mat.* **10** (1972), 59–77.
30. V. Totik, Distribution of simple zeros of polynomials, *Acta Math.* **170** (1993), 1–28.
31. M. Tsuji, "Potential Theory in Modern Function Theory," Chelsea, New York, 1950.